(2) I have only a partial solution for this question: if $b = 0$ and $\sqrt{-\frac{c}{a}} \in \mathbb{Q} \setminus \mathbb{N}$. In this case:

$$S = \sum_{n=1}^{+\infty} \frac{1}{p^2 n^2 - q^2} \qquad \text{with} \qquad \frac{q}{p} \notin \mathbb{N}.$$

By the expansion of the Fourier series of the function $x \mapsto \cos \frac{qx}{p}$ over $[-\pi, \pi]$ we have

$$S = \frac{1}{2q^2} - \frac{\pi}{2pq \tan \frac{\pi q}{q}}.$$

Assume, by contradiction, that $S \in \mathbb{Q}$, then $\tan \frac{\pi q}{p} = \pi\lambda$ with $\lambda \in \mathbb{Q}$. By De Moivre's formula we have

$$\sum_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} \binom{p}{2k+1} (-1)^k (\pi\lambda)^{2k+1} \quad = \quad 0.$$

Since $\lambda \in \mathbb{Q}$ and π is a transcendental number, this is impossible.

Editor's Comment: Moubinool Omarjee pointed out that this problem had been previously proposed by R. André-Jeannin, which appeared in *RMS Revue Mathématiques Spéciales Q79: pp. 203-204; 1987-1988.*

Also solved by Moubinool Omarjee, Lycée Henri IV, Paris, France.

79. *Proposed by Mihály Bencze, Brasov, Romania.* If $x, y, z \in \mathbb{R}$, Prove that:

(1) $2\Big(|\cos x| + |\cos y| + |\cos z|\Big) + |\cos(y+z)| + |\cos(x+z)| + |\cos(y+x)| \geq 3$
(2) $|\cos x| + |\cos y| + |\cos z| + |\cos(y+z)| + |\cos(y+x)| + |\cos(x+z)| + 3|\cos(x+y+z)| \geq 3$.

Solution 1 by Arkady Alt, San Jose, California, USA. First we will prove that for any $x, y \in \mathbb{R}$the following inequality holds

$$|\cos x| + |\cos y| + |\cos(x+y)| \geq 1. \tag{1}$$

Since

$$|\cos(x+y)| = |\cos x \cos y - \sin x \sin y| \geq ||\cos x| \, |\cos y| - |\sin x| \, |\sin y||$$

it suffices to prove the inequality

$$||\cos x| \, |\cos y| - |\sin x| \, |\sin y|| \geq 1 - |\cos x| - |\cos y|$$

or equivalently

$$\left| uv - \sqrt{1-u^2} \cdot \sqrt{1-v^2} \right| \geq 1 - u - v, \quad \text{where } u, v \in [0,1]. \tag{2}$$

Inequality (2) holds since

$$\left| uv - \sqrt{1-u^2} \cdot \sqrt{1-v^2} \right| \geq \sqrt{1-u^2} \cdot \sqrt{1-v^2} - uv$$

and

$$\sqrt{1-u^2} \cdot \sqrt{1-v^2} - uv \geq 1 - u - v \iff \sqrt{1-u^2} \cdot \sqrt{1-v^2} \geq (1-u)(1-v)$$
$$\iff \sqrt{1-u}\sqrt{1-v}\left(\sqrt{1+u}\sqrt{1+v} - \sqrt{1-u}\sqrt{1-v}\right) \geq 0.$$

(a) Applying inequality (1) we obtain

$$\sum_{\text{cyclic}} \left(|\cos x| + |\cos y| + |\cos (x+y)|\right) \geq \sum_{cyclic} 1 = 3,$$

which is equivalent to first inequality.
(b) Let $\alpha = x, \beta = y + z$ then by (1) we have

$$|\cos x| + |\cos (y+z)| + |\cos (x+y+z)| = |\cos \alpha| + |\cos \beta| + |\cos (\alpha + \beta)| \geq 1,$$

and therefore,

$$\sum_{\text{cyclic}} \left(|\cos x| + |\cos (y+z)| + |\cos (x+y+z)|\right) \geq 3,$$

which is equivalent to the second inequality.

Solution 2 by Moti Levy, Rehovot, Israel. Define the piecewise linear π-periodic function f, by

$$f(x) = \begin{cases} 1 - \frac{2}{\pi}x & \text{if } \; 0 \leq x \leq \frac{\pi}{2}, \\ \frac{2}{\pi}x - 1 & \text{if } \; \frac{\pi}{2} \leq x \leq \pi, \end{cases}$$

By definition, $f(x)$ satisfies

$$f(x) \leq |\cos x|.$$

The function $x \mapsto |\cos x|$ is periodic with period π, hence we may consider only $0 \leq x, y, z \leq \pi$.
We will write $F(x, y, z)$ and $G(x, y, z)$ to denote, respectively,

$$2\Big(|\cos x| + |\cos y| + |\cos z|\Big) + |\cos(y+z)| + |\cos(x+z)| + |\cos(y+x)|$$

and

$$|\cos x| + |\cos y| + |\cos z| + |\cos(y+z)| + |\cos(y+x)| + |\cos(x+z)| + 3|\cos(x+y+z)|$$

Since the inequalities are symmetric in the variables x, y, z and the function $|\cos x|$ is symmetric with respect to $\frac{\pi}{2}$, it is sufficient to consider only the following cases:
Case 1: $x, y, z \leq \frac{\pi}{2},\ x + y \leq \frac{\pi}{2},\ x + z \leq \frac{\pi}{2},\ y + z \leq \frac{\pi}{2}$
It follows that $x + y + z \leq \frac{3\pi}{4}$.

$$\begin{aligned} F(x, y, z) &\geq 2\left(1 - \frac{2x}{\pi} + 1 - \frac{2y}{\pi} + 1 - \frac{2z}{\pi}\right) \\ &\quad + 1 - \frac{2}{\pi}(x+y) + 1 - \frac{2}{\pi}(y+z) + 1 - \frac{2}{\pi}(x+z) \\ &= 9 - \frac{8}{\pi}(x+y+z) \geq 9 - \frac{1}{\pi} 8 \times 3 \times \frac{\pi}{4} = 3. \end{aligned}$$

If $x + y + z \leq \frac{\pi}{2}$ then

$$\begin{aligned} G(x, y, z) &\geq 1 - \frac{2x}{\pi} + 1 - \frac{2y}{\pi} + 1 - \frac{2z}{\pi} \\ &\quad + 1 - \frac{2}{\pi}(x+y) + 1 - \frac{2}{\pi}(y+z) + 1 - \frac{2}{\pi}(x+z) + 3\left(1 - \frac{2}{\pi}(x+y+z)\right) \\ &= 9 - \frac{12}{\pi}(x+y+z) \geq 9 - \frac{12}{\pi} \times \frac{\pi}{2} = 3, \end{aligned}$$